C.T. Conley and B.D. Miller. *A bound on measurable chromatic numbers of locally finite Borel graphs.* ***Mathematical Research Letters***, vol. 23 no. 6 (2016), pp. 1633–1644.

This article is a nice and short contribution to *descriptive combinatorics*, dealing with colorings of a *locally finite* Borel graph G on a Polish space X. A priori, even with the local finiteness hypothesis, the abstract chromatic number $\chi(G)$ of G may be much smaller than its analogues with additional regularity conditions on the coloring function (e.g. measurability). The article provides an upper bound of $2\chi(G)-1$ for measurable and Baire measurable chromatic numbers for hyperfinite graphs.

Being comfortable with basic measure theory and Baire category theory should be enough to understand the core parts of the paper; in particular, it should be accessible to many graduate students, including those not familiar with descriptive set theory.

The subject of *descriptive combinatorics* was founded in the seminal paper of Kechris, Solecki, and Todorčević, *Borel chromatic numbers*, ***Adv. in Math.***, vol. 141 (1999), pp. 1–44. It studies definable (most often Borel or analytic) graphs on Polish spaces and its broad goal is determining the extent to which the usual combinatorial notions such as colorings and matchings behave under definability constraints; see Kechris S. and Marks A., *Descriptive graph combinatorics*, http://math.caltech.edu/~kechris/papers/combinatorics16.pdf, for a comprehensive state-of-the-art survey.

By a *Borel* (resp. *analytic*, *co-analytic*) *graph* G on a Polish space X we mean an irreflexive and symmetric Borel (resp. analytic, co-analytic) subset of X^2. Such graphs naturally appear all over descriptive set theory, analysis, measurable and topological dynamics, and other areas. Analytic graphs most often arise as unions of graphs of functions; for example, a Borel action $\alpha : \Gamma \curvearrowright X$ of a Polish group Γ and an analytic symmetric set $V \subseteq \Gamma$ induces the analytic graph by putting an edge between $x, y \in X$ if and only if $y = \gamma \cdot x$ for some $\gamma \in \Gamma$; this graph would be Borel if V is countable. Interesting examples of different nature include the graph on X of being within distance 1, as well as the intersection graph on the (standard Borel) space $[X]^{<\infty}$ of all finite nonempty subsets of X.

For a Polish space Y, a Y*-coloring* of G is a function $c : X \to Y$ that assigns distinct values to G-adjacent vertices. The *abstract* (resp. *Borel*, *Baire measurable*, μ*-measurable*—for a Borel measure μ on X) *chromatic number* of G, denoted by $\chi(G)$ (resp. $\chi_{\mathrm{B}}(G)$, $\chi_{\mathrm{BM}}(G)$, $\chi_{\mu}(G)$) is the least cardinality of Y for which a (resp. Borel, Baire measurable, μ-measurable) Y-coloring exists. Imposing stronger definability conditions on the coloring function, thus requiring it to be more constructive, may significantly increase the chromatic number. Example 3.1 of the aforementioned paper of Kechris, Solecki, and Todorčević provides an acyclic locally countable Borel graph G whose Baire measurable and μ-measurable (and hence also Borel) chromatic numbers are $2^{\aleph_0}$, whereas Axiom of Choice allows to build a 2-coloring by starting from a chosen point in each connected component simultaneously.

Even for graphs whose degree is bounded by some $d \geq 1$, the situation is still very interesting. The abstract chromatic number of such graphs is at most $d+1$: any partial coloring has a proper extension (there is always one extra color to resolve any conflict), so once some vertices are colored in every connected component, induction takes care of the rest. It is shown in the above-mentioned paper of Kechris, Solecki, and Todorčević that one can run the same algorithm in a Borel fashion, so the Borel chromatic number of such graphs is also at most $d+1$. However, unlike in the abstract case, this bound is sharp even for acyclic graphs: a striking result of Marks in his article *A determinacy approach to Borel combinatorics*, ***J. Amer. Math. Soc.***, vol. 29 (2016), pp. 579–600, states that for any $d \geq 1$, there is an acyclic d-regular Borel graph G with $\chi_{\mathrm{B}}(G) = d+1$.

The gap between the Borel chromatic number and the abstract chromatic number is not just in seen in acyclic graphs. Indeed, a classical theorem of Brooks states that any graph of degree at most $d \geq 3$ admits a d-coloring, unless it contains a d-clique. In contrast with Borel colorings, Conley, Marks, and Tucker-Drob proved in their paper *Brooks' theorem for measurable colorings*, ***Forum of Mathematics, Sigma***, vol. 4, that the complete analogue of Brooks' theorem holds for Baire measurable and μ-measurable colorings.

Turning now to the article under the present review, we state its results:

1. **Probability measure setting:** Let a Polish space X be equipped with a Borel probability measure μ. For any analytic hyperfinite graph G on X (i.e. G is an increasing union of analytic graphs with finite connected components),
$$\chi_\mu(G) \leq 2\chi(G) - 1.$$
2. **Baire category setting:** For any analytic graph G on a Polish space X,
$$\chi_{\mathrm{BM}}(G) \leq 2\chi(G) - 1.$$

We highlight that, unlike Brooks' theorem, these results apply to all hyperfinite locally finite Borel (even analytic) graphs, whose degree is not necessarily bounded. And even for a hyperfinite graph G whose degree is bounded by $d \geq 1$ and $2\chi(G)-1 < d$, these results provide a substantial improvement on Brooks' theorem; e.g. if G is acyclic then $2\chi(G) - 1 = 3 \ll d$.

Points to note:

- Although the result for the Baire measurable chromatic number, unlike for the μ-measurable one, is stated and proven without the assumption of hyperfiniteness, adding this assumption causes no loss of generality because every locally countable Borel graph is hyperfinite off of a meager set; see Theorem 6.2 in Hjorth G. and Kechris A., *Borel equivalence relations and classifications of countable models*, ***Ann. Pure Appl. Logic***, vol. 82 (1996), no. 3, 221–272.
- For the Baire measurable (resp. μ-measurable) chromatic number, the upper bound of $2\chi(G)-1$ is typically sharp for the graphs G induced by generically ergodic (resp. μ-ergodic) Borel actions of $\mathbb{Z}$ (in particular, each connected component of G is a line).
- The assumption of hyperfiniteness cannot be dropped from the statement about μ-measurable chromatic numbers as Corollary 0.8 of Conley C. and Kechris A., *Measurable chromatic and independence numbers for ergodic graphs and group actions*, ***Groups Geom. Dyn.***, vol. 7 (2013), no. 1, pp. 127–180, provides, for each $n \geq 2$, an example of an acyclic (measure-preserving) Borel graph G of bounded degree with $\chi_\mu(G) = n$.

Both of the results are first proven for Borel graphs and then generalized to analytic graphs via classical descriptive-set-theoretic techniques such as the *first reflection principle*. Assuming the graph G is Borel, the idea behind the proofs of both of the results is as follows. First, we remark that if a Borel graph G is *component-finite* (i.e. each of its connected components are finite), then $\chi_{\mathrm{B}}(G) = \chi(G)$; indeed, because each connected component has only finitely many possible $\chi(G)$-colorings, the Luzin–Novikov uniformization allows for picking one for each connected component in a uniformly Borel fashion. Letting now G be a locally finite Borel graph on a Polish space X, call a set $B \subseteq X$ a *G-barrier* if removing B from X makes $G|_{X \setminus B}$ component-finite. In the Baire measurable (resp. μ-measurable) setting, ignoring a meager (resp. μ-null) set, the authors construct a Borel set $B \subseteq X$ such that both B and $X \setminus B$ are G-barriers;

in particular, $G|_B$ admits a Borel $\chi(G)$-coloring c_0. Furthermore, B is such that even after removing one of the colors from B, the remaining set B' is still a G-barrier, so there is a Borel $\chi(G)$-coloring c_1 of $G|_{X \setminus B'}$. But then the disjoint union of $c_0|_{B'}$ and c_1 is a Borel $(2\chi(G) - 1)$-coloring of the entire space X.

The construction of such a set B is "random" in both settings. In the Baire measurable setting, the authors use the Kuratowski–Ulam method: for each parameter $\alpha \in \mathbb{N}^{\mathbb{N}}$, they give a recipe for building a set B_α and show (swapping the quantifiers) that for comeagerly-many parameters $\alpha \in \mathbb{N}^{\mathbb{N}}$, B_α has the desired properties on a comeager subset of X. In the μ-measurable setting, the exchange of quantifiers is done via the Borel–Canteli lemma, which allows for a more direct construction of the set B.

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